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Proof of the convexity of the function $L(\sigma, T)$ for small T and the Riemann hypothesis

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Abstract. The authors of this article prove the Riemann hypothesis for a special case when the real σ and imaginary $T = 1/(2a)$ parts of the zeta function argument belong to a certain region Θ_R of the complex plane. It is based on V. Blinovskiy's method: 1) an expression is derived for the function $L(\sigma, a)$, the square of the modulus of the fundamental part of the zeta function, which affects its zeros, 2) the proof of the Riemann hypothesis is reduced to proving the U-convexity of the function L by σ , i. e. the positivity of $L_{\sigma\sigma}^{(2)}(\sigma, a)$. The authors provide estimates that allow us to describe the region Θ_R , and present the data of a computational experiment conducted in the Maxima mathematical package by constructing the region Θ_R . A literature review on the current state of hypothesis proof is provided. Further development of the method is proposed both in terms of expanding the region Θ_R and for increasing computational efficiency.

Keywords: Riemann hypothesis, Blinovskiy's method, zeta function, gamma function, Maxima mathematical package

Mathematics Subject Classification: 11M26.

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НАУЧНАЯ СТАТЬЯ

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Доказательство выпуклости функции $L(\sigma, T)$ при малых T и гипотеза Римана

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Аннотация. В статье доказывается гипотеза Римана для частного случая, когда действительная σ и мнимая $T = 1/(2a)$ части аргумента дзета-функции принадлежат определенной области Θ_R комплексной плоскости. В основу положен метод В. Блиновского: 1) выводится выражение для функции $L(\sigma, a)$ — квадрата модуля основной части дзета-функции, влияющей на ее нули, 2) доказательство гипотезы Римана сводится к доказательству выпуклости вниз функции L по σ , т. е. положительности $L_{\sigma\sigma}^{(2)}(\sigma, a)$. Авторами показаны оценки, позволяющие описать область Θ_R , приведены данные вычислительного эксперимента, проведенного в математическом пакете *Mathima*, по построению области Θ_R . Сделан обзор литературы по современному состоянию вопроса доказательства гипотезы. Предложено дальнейшее развитие метода как с точки зрения расширения области Θ_R , так и для повышения вычислительной эффективности.

Ключевые слова: гипотеза Римана, метод Блиновского, дзета-функция, гамма-функция, математический пакет *Mathima*

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Introduction

Let us consider the representation [1, p. 1045], [2, p. 20] [3, p. 61], [4, 5] of the Riemann zeta function

$$\zeta(z) = \frac{I(z)}{\Gamma(z)(1 - 2^{1-z})}, \quad I(z) = \int_0^\infty \frac{x^{z-1}}{e^x + 1} dx, \quad (0.1)$$

where¹⁾ $z = \sigma + iT$, $\sigma > 0$, at $\sigma = 1$ $T \neq \frac{2\pi}{\ln 2}m$, $m \in \mathbf{Z}$ (unfortunately, this fact is not mentioned in many mathematical reference books).

The formula (0.1) is obtained from the standard representation

$$\zeta(z) = \frac{1}{\Gamma(z)} \int_0^\infty \frac{x^{z-1}}{e^x - 1} dx \quad \text{for } \sigma > 1,$$

as:

1. The following identity is used

$$\frac{1}{e^x + 1} = \frac{1}{e^x - 1} - \frac{2}{e^{2x} - 1}. \quad (0.2)$$

2. Let us multiply the both sides of the equality (0.2) by x^{z-1} and $1/\Gamma(z)$ and integrate it on x :

$$\frac{1}{\Gamma(z)} \int_0^\infty \frac{x^{z-1}}{e^x + 1} dx = \frac{1}{\Gamma(z)} \int_0^\infty \frac{x^{z-1}}{e^x - 1} dx - \frac{1}{2^{z-1}} \frac{1}{\Gamma(z)} \int_0^\infty \frac{(2x)^{z-1}}{e^{2x} - 1} d(2x),$$

from which we have the equality

$$\frac{1}{\Gamma(z)} \int_0^\infty \frac{x^{z-1}}{e^x + 1} dx = \zeta(z) - 2^{1-z} \zeta(z).$$

This equality gives the formula (0.1).

3. We have the inequality

$$|I(z)| \leq \int_0^\infty x^{\sigma-1} e^{-x} dx = \Gamma(\sigma),$$

which makes sense at $\sigma > 0$, and

$$1 - 2^{1-z} \neq 0 \quad \text{or} \quad e^{(1-z)\ln 2} \neq e^{2\pi ki},$$

where $k \in \mathbf{Z}$, from which we obtain

$$z \neq 1 + \frac{2\pi(-k)}{\ln 2}i,$$

let $m = -k$.

¹⁾The ambiguity of the function x^{z-1} is usually eliminated at defining this function as $x^{z-1} = e^{(z-1)\ln x}$ where the logarithm $\ln x$ is real (for example, see [3, p. 22]).

For $\sigma < 0$ the zeta function is extended using the Riemann functional equation [2, p. 19] from the ratio

$$\zeta(z) = 2(2\pi)^{z-1} \sin\left(\frac{\pi}{2}z\right) \Gamma(1-z)\zeta(1-z). \quad (0.3)$$

The formula (0.3) is also valid for $\sigma \in [0, 1]$. At $\sigma = 0$ the value $\zeta(iT)$ can be expressed through $\zeta(1-iT)$ from (0.3).

In the preprint [6], the author presents a linear non-homogeneous ordinary differential equation of infinite order with linear coefficients by z (including the highest derivative in the general form of the series) for the Hurwitz zeta function

$$\zeta(z, a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^z}$$

(which is a generalization of the Riemann zeta function) that can be used to find approximate values of this function using, for example, numerical schemes [7]. In this case, the coefficients of the derivatives become unknown functions, and the resulting system of equations for the original equation with truncated order will have quadratic nonlinearities. Note that an effective numerical method for calculating the value of the zeta function is considered in the article [8].

The trivial zeros of the zeta function are $z = -2n$, $n \in \mathbf{N}$ (it follows from ratio (0.3)).

From the definition

$$\zeta(z) = 1 + \sum_{n=2}^{\infty} \frac{1}{n^z}, \quad \sigma > 1,$$

it follows that $\zeta(z)$ has no zeros when $\sigma > 1$ and $\zeta(z)$ has only the trivial zeros at $\sigma < 0$. The possible zeros of the zeta function in the strip $\sigma \in [0, 1]$ are called nontrivial. Note that the non-trivial zeros are also absent on the line $\sigma = 1$. The various proofs is given in the book [9, p. 50–54].

In 1859, Riemann put forward the hypothesis that all nontrivial zeroes of $\zeta(z)$ are concentrated on the line $\sigma = 1/2$. Some modern work indicate the boundaries of the regions in which the absence of zeros of zeta function has been proven. For example, S. Stechkin [10] proved the Riemann hypothesis at

$$|T| \geq 12, \quad \sigma \geq 1 - \frac{1}{R \ln |T|},$$

where $R = 9.65$.

The main method for studying the zeta function today is the Vinogradov's method [11].

A. Karatsuba [12, p. 75–77] proved the theorem that there exists a constant $c > 0$ such that the function $\zeta(z) \neq 0$ in the region

$$|T| \geq 10, \quad \sigma \geq 1 - \frac{c}{\ln^{2/3} |T| \ln \ln |T|}.$$

Using the methods of N. Korobov [13] and I. Vinogradov, K. Ford [14] proved the Riemann hypothesis, for example, at

$$|T| \geq 3, \quad \sigma \geq 1 - \frac{c}{\ln^{2/3} |T| \ln^{1/3} \ln |T|},$$

where $c = 1/57.54$.

The result of Stechkin was later [15] improved: $|T| \geq 3$, $R = 8.464$.

Today, the largest known zero-free region Ω_R for the zeta-function is at $|T| \geq 2$ and $R = 5.573412$ [16] (we don't take the computational experiments into account). This result improves on the known relations for $3 \cdot 10^{12} \leq |T| \leq e^{64.1}$ and $|T| \geq e^{1000}$ [17].

Based on fast algorithms [18] for multiple calculations of the Riemann zeta function, its zeros were found up to $|T| \leq 1.3 \cdot 10^{21}$ [19]. Using condition (4) in the manuscript [20], V. Blinovskiy conclude that from the assumption of the convexity of function $L(\sigma, a)$ over σ (see below the definition of $L(\sigma, a)$) in the region

$$|T| > 1.3 \cdot 10^{21}, \quad \sigma \in \left(\frac{1}{60 \ln |T|}, 1 - \frac{1}{60 \ln |T|} \right)$$

follows the Riemann hypothesis.

In this article we make some step in the determining the region in $\Psi_{R1} \subset ((0, 1), \mathbf{R})$, where the function $L(\sigma, a)$ is convex on σ . At the same time, $\Psi_{R1} \cap \Omega_R = \emptyset$. We show the computer program and result of the computational experiment.

1 The Square of the Modulus of the Function $I(z)$

Let us move from the function $I(z)$ to the function $L(\sigma, a)$.

From the formula (0.3), if $z = z^*$ ($\sigma^* \in (0, 1)$) is non-trivial zero of $\zeta(z)$, then $z = 1 - z^*$ is also non-trivial zero.

Note from the formula (0.1) that the nontrivial zeroes of $\zeta(z)$ are influences only by the zeroes of $I(z)$. Therefore, further all reasoning will be carried out for the function $I(z)$.

It is easy to show

$$\overline{I(z)} = I(\bar{z}).$$

Now we will consider the function $K(\sigma, T)$ that is the square of the function $I(z)$:

$$\begin{aligned} K(\sigma, T) &= |I(z)|^2 = \frac{1}{2} (I(z)I(\bar{z}) + I(\bar{z})I(z)) \\ &= \frac{1}{2} \left(\int_0^\infty \int_0^\infty \frac{x^{z-1}y^{\bar{z}-1}}{(e^x + 1)(e^y + 1)} dx dy + \int_0^\infty \int_0^\infty \frac{x^{\bar{z}-1}y^{z-1}}{(e^x + 1)(e^y + 1)} dx dy \right) \\ &= \frac{1}{2} \int_0^\infty \int_0^\infty \frac{x^{z-1}y^{\bar{z}-1} + x^{\bar{z}-1}y^{z-1}}{(e^x + 1)(e^y + 1)} dx dy. \end{aligned} \quad (1.1)$$

Let us write numerator

$$\begin{aligned} x^{z-1}y^{\bar{z}-1} + x^{\bar{z}-1}y^{z-1} &= x^{\sigma+iT-1}y^{\sigma-iT-1} + x^{\sigma-iT-1}y^{\sigma+iT-1} \\ &= (xy)^{\sigma-1} (x^{iT}y^{-iT} + x^{-iT}y^{iT}) = (xy)^{\sigma-1} \left(e^{iT \ln \frac{x}{y}} + e^{-iT \ln \frac{x}{y}} \right) \\ &= 2(xy)^{\sigma-1} \cos \left(T \ln \frac{x}{y} \right). \end{aligned}$$

Let us make a substitution $y = \frac{c^2}{x}$ in the integral (1.1), $dy = \frac{2c}{x} dc$:

$$K(\sigma, T) = 2 \int_0^\infty \int_0^\infty \frac{c^{2\sigma-1} \cos \left(2T \ln \frac{x}{c} \right)}{x(e^x + 1)(e^{c^2/x} + 1)} dx dc.$$

Next we will introduce one more substitution $u = 2T \ln \frac{x}{c}$, $dx = ce^{u/(2T)} \frac{1}{2T} du$, at $T > 0$:

$$K(\sigma, T) = \frac{1}{T} \int_{-\infty}^{\infty} \int_0^{\infty} \frac{c^{2\sigma-1} \cos u}{(e^{ce^{u/(2T)}} + 1)(e^{ce^{-u/(2T)}} + 1)} dudc, \quad (1.2)$$

at $T < 0$:

$$K(\sigma, T) = \frac{1}{T} \int_{-\infty}^{\infty} \int_0^{\infty} \frac{c^{2\sigma-1} \cos u}{(e^{ce^{u/(2T)}} + 1)(e^{ce^{-u/(2T)}} + 1)} dudc. \quad (1.3)$$

In the integral (1.3) we introduce the substitution $u = -w$:

$$K(\sigma, T) = \frac{1}{-T} \int_{-\infty}^{\infty} \int_0^{\infty} \frac{c^{2\sigma-1} \cos w}{(e^{ce^{w/(2(-T))}} + 1)(e^{ce^{-w/(2(-T))}} + 1)} dw dc,$$

which is analogous to the integral (1.2).

Then it is enough to consider the proof for $T > 0$.

Taking into account the parity of the integrand in (1.2) with respect to the variable u , we rewrite the integral as

$$K(\sigma, 1/(2a)) = 4aL(\sigma, a), \quad (1.4)$$

$$L(\sigma, a) = \int_0^{\infty} \int_0^{\infty} \frac{y^{2\sigma-1} \cos x}{(e^{ye^{ax}} + 1)(e^{ye^{-ax}} + 1)} dx dy \geq 0,$$

where $a = \frac{1}{2T} > 0$.

If $z = z^*$ is non-trivial zero of $\zeta(z)$, then $\sigma_1^* = \sigma^*$ and $\sigma_2^* = 1 - \sigma^*$ is minima of the function $L(\sigma, a)$ in which it is equal to zero in the interval $(0, 1)$. Then from its continuity (will be proven later) and the formula (1.4) it follows that it changes the direction of convexity. Therefore, to prove the Riemann hypothesis ($\sigma_1^* = \sigma_2^* = 1/2$), it is necessary to show the U-convexity of the function $L(\sigma, a)$ on σ at $(\sigma, T) \in \Psi_{R1}$.

We have

$$L_{\sigma\sigma}^{(2)}(\sigma, a) = 4 \int_0^{\infty} \int_0^{\infty} \frac{y^{2\sigma-1} \ln^2 y \cos x}{(e^{ye^{ax}} + 1)(e^{ye^{-ax}} + 1)} dx dy. \quad (1.5)$$

Note that at

$$x \in \left[\frac{\pi}{2}(2n+1), \frac{\pi}{2}(2n+3) \right], \quad n \in \mathbf{N},$$

we have $\cos x \geq 0$ and at

$$x \in \left[\frac{\pi}{2}(2n+1), \frac{\pi}{2}(2n+3) \right], \quad n \in \mathbf{N}$$

we have $\cos x \leq 0$.

Because the functions $e^{ax} + e^{-ax}$ and $e^{ye^{ax}} + e^{ye^{-ax}}$ are increasing $a > 0$, the function

$$(e^{ye^{ax}} + 1)(e^{ye^{-ax}} + 1) = e^{y(e^{ax} + e^{-ax})} + e^{ye^{ax}} + e^{ye^{-ax}} + 1 \quad (1.6)$$

is increasing at $x \geq 0$ and $y \geq 0$ also and hence

$$\varphi(x, y; \sigma, a) = \frac{1}{(e^{ye^{ax}} + 1)(e^{ye^{-ax}} + 1)} \downarrow$$

at $x \geq 0$ and $y \geq 0$.

Hence area of under the function $\varphi(x, y; \sigma, a) \cos(x)$, $x \in [\frac{\pi}{2}(2n+1), \frac{\pi}{2}(2n+3)]$ is greater than the area above the function $\varphi(x, y; \sigma, a) \cos(x)$, $x \in [\frac{\pi}{2}(2n+1), \frac{\pi}{2}(2n+3)]$, $n \in \mathbf{N}$.

Thus we have

$$\begin{aligned} I^*(\sigma, a) &= \int_{3\pi/2}^{\infty} \int_0^{\infty} y^{2\sigma-1} \ln^2 y \varphi(x, y; \sigma, a) \cos x \, dy dx \\ &= \sum_{n=1}^{\infty} \left(\int_{(2n+1)\pi/2}^{(2n+3)\pi/2} \int_0^{\infty} y^{2\sigma-1} \ln^2 y \varphi(x, y; \sigma, a) \cos x \, dy dx \right. \\ &\quad \left. + \int_{(2n+3)\pi/2}^{(2n+5)\pi/2} \int_0^{\infty} y^{2\sigma-1} \ln^2 y \varphi(x, y; \sigma, a) \cos x \, dy dx \right) > 0. \end{aligned}$$

Hence to prove that $L_{\sigma\sigma}^{(2)}(\sigma, a) \geq 0$ for (σ, T) in some region from $(0, 1) \times \mathbf{R}$ it is sufficient to show the validity in this area of the inequality

$$\int_0^{3\pi/2} \int_0^{\infty} y^{2\sigma-1} \ln^2 y \varphi(x, y; \sigma, a) \cos x \, dy dx \geq 0.$$

Denote

$$\begin{aligned} I_1(\sigma, a) &= \int_0^{\pi/2} \int_0^{\infty} y^{2\sigma-1} \ln^2 y \varphi(x, y; \sigma, a) \cos x \, dy dx, \\ I_2(\sigma, a) &= \int_{\pi/2}^{3\pi/2} \int_0^{\infty} y^{2\sigma-1} \ln^2 y \varphi(x, y; \sigma, a) \cos x \, dy dx. \end{aligned}$$

Because $\cos x \geq 0$, $x \in [0, \pi/2]$, $\cos x \leq 0$, $x \in [\pi/2, 3\pi/2]$, we have $I_1(\sigma, a) > 0$ and $I_2(\sigma, a) < 0$.

Define the region

$$\Psi_{R1} = \left\{ (\sigma, T) : I_1(\sigma, a) + I_2(\sigma, a) = \int_0^{3\pi/2} \int_0^{\infty} y^{2\sigma-1} \ln^2 y \varphi(x, y; \sigma, a) \cos x \, dy dx \geq 0 \right\}.$$

We have

$$\begin{aligned} I_1(\sigma, a) + I_2(\sigma, a) &= \int_0^{\pi/2} \int_0^{\infty} y^{2\sigma-1} \ln^2 y \varphi(x, y; \sigma, a) \cos x \, dy dx \\ &\quad + \int_{\pi/2}^{3\pi/2} \int_0^{\infty} y^{2\sigma-1} \ln^2 y \varphi(x, y; \sigma, a) \cos x \, dy dx. \end{aligned}$$

From (1.6)

$$(e^{ye^{ax}} + 1)(e^{ye^{-ax}} + 1) \leq 4e^{y(e^{ax} + e^{-ax})}$$

and

$$(e^{ye^{ax}} + 1)(e^{ye^{-ax}} + 1) \geq e^{y(e^{ax} + e^{-ax})},$$

then

$$\frac{1}{4}e^{-y(e^{ax} + e^{-ax})} \leq \varphi(x, y; a) \leq e^{-y(e^{ax} + e^{-ax})}. \quad (1.7)$$

In the integral

$$g(x; \sigma, a) = \int_0^{\infty} y^{2\sigma-1} e^{-y(e^{ax} + e^{-ax})} dy$$

we make substitution $t = y(e^{ax} + e^{-ax})$. Then $y = \frac{t}{e^{ax} + e^{-ax}}$, $dy = \frac{dt}{e^{ax} + e^{-ax}}$.

We get

$$g(x; \sigma, a) = \int_0^\infty \frac{t^{2\sigma-1} e^{-t}}{(e^{ax} + e^{-ax})^{2\sigma}} dt = \frac{\Gamma(2\sigma)}{(e^{ax} + e^{-ax})^{2\sigma}}$$

and, thus

$$\frac{g(x; \sigma, a)}{4} \leq \int_0^\infty y^{2\sigma-1} \varphi(x; \sigma, a) dy \leq g(x; \sigma, a).$$

Then the integral (1.4) is convergent, the function $L(\sigma, a)$ is continuous in the region Ψ_{R1} [21, p. 501].

By analogy, we estimate the integral (1.5). Let

$$\begin{aligned} A(x; \sigma, a) &= \int_0^\infty y^{2\sigma-1} \ln^2 y e^{-y(e^{ax} + e^{-ax})} dy \\ &= \frac{1}{(e^{ax} + e^{-ax})^{2\sigma}} (\Gamma^{(2)}(2\sigma) - 2\Gamma^{(1)}(2\sigma) \ln(e^{ax} + e^{-ax}) + \Gamma(2\sigma) \ln^2(e^{ax} + e^{-ax})) \end{aligned}$$

or

$$A(x; \sigma, a) = \frac{\Gamma(2\sigma)}{(e^{ax} + e^{-ax})^{2\sigma}} \left(\frac{\Gamma^{(2)}(2\sigma)}{\Gamma(2\sigma)} + \ln(e^{ax} + e^{-ax}) \left(\ln(e^{ax} + e^{-ax}) - 2 \frac{\Gamma^{(1)}(2\sigma)}{\Gamma(2\sigma)} \right) \right).$$

Then the integral (1.5) is convergent.

From the inequalities (1.7), the region Ψ_{R1} is defined by the couples (σ, T) such that the following inequality is satisfied

$$h_1(\sigma, a) = \frac{1}{4} \int_0^{\pi/2} A(x; \sigma, a) \cos x dx + \int_{\pi/2}^{3\pi/2} A(x; \sigma, a) \cos x dx \geq 0.$$

Let

$$\Psi_{R2} = \{(\sigma, T) : (1 - \sigma, T) \in \Psi_{R1}\}.$$

From the formula (0.3), the Riemann hypothesis is true for $\Theta_R = \Psi_{R1} \cup \Psi_{R2}$.

2 The Computational Experiment

The following program developed in the package Maxima [22] allows to construct the region Θ_R . Maxima supports the calculation of both the first derivative of the gamma function and the second derivative.

```
diff(gamma(x), x, 1);
dgamma(x) := float(' %);
diff(gamma(x), x, 2);
d2gamma(x) := float(' %);
A(x, sigma, a) := float(gamma(2*sigma) / (exp(a*x) + exp(-a*x))^(2*sigma) *
    (d2gamma(2*sigma) / gamma(2*sigma) + log(exp(a*x) + exp(-a*x)) *
    (log(exp(a*x) + exp(-a*x)) - 2*dgamma(2*sigma) / gamma(2*sigma))));
s1: 0.01;
s2: 0.99;
T1: 0.01;
T2: 1;
```

```

l:[];
delta_sigma:0.001;
delta_T:0.001;
for sigma:s1 step delta_sigma thru s2 do
(
  for T:T1 step delta_T thru T2 do
  (
    a:1/(2*T),
    h:0.25*quad_qags(A(x,sigma,a)*cos(x),x,0,%pi/2)[1]+
      quad_qags(A(x,sigma,a)*cos(x),x,%pi/2,3*%pi/2)[1],
    if h >= 0 then (l:append(l,[[sigma,T]]), l:append(l,[[1-sigma,T]]))
  )
);
load(draw);
draw2d(file_name="h:\mygraph1",terminal='png,
  font="Arial",font_size=18,
  dimensions=[1500,1500],
  xlabel = "sigma", ylabel = "T",
  point_type=filled_circle,
  point_size=1,color=blue,
  points(l));

```

In this program, it set the search boundaries and the grid interval by σ and T : $\sigma_1 = 0.01$, $\sigma_2 = 0.99$, $\sigma \in [\sigma_1, \sigma_2]$, $\delta\sigma = 0.001$ (the grid step by σ), $T_1 = 0.01$, $T_2 = 1$, $T \in [T_1, T_2]$, $\delta T = 0.001$ (the grid step by T), `quad_qags()` is a numerical integration of a general function over a finite interval by the Epsilon algorithm in Maxima. Fig. 1 shows the constructed region Θ_R (it painted over with color).

The computing time is about 2 hours on the processor AMD Ryzen 7 5700G. The problem of reducing computing time can be solved well using parallel computing. To organize such an enumeration of values, the grid layers on σ are divided into approximately equal subsets, and each such subset analyzes a separate computational process, for example, within a computing cluster.

Conclusion

The article proposes a method for constructing the region Ψ_{R1} where the function $L(\sigma, a)$ is convex, and it follows that in the region Ψ_{R1} the function $\zeta(z)$ are no zeros (therefore, in the region Θ_R), and presents the results of a computational experiment. Further expansion of this region can be achieved by extracting the positive sums from the integral $I^*(\sigma, a)$, i. e., for example, considering the positivity of the function

$$h_2(\sigma, a; N) = h_1(\sigma, a) + \sum_{n=1}^N \left(\frac{1}{4} \int_{(2n+1)\pi/2}^{(2n+3)\pi/2} A(x; \sigma, a) \cos x \, dx + \int_{(2n+3)\pi/2}^{(2n+5)\pi/2} A(x; \sigma, a) \cos x \, dx \right).$$

For example, for $N = 1$ the region Θ_R is shown in Fig. 2, $T_2 = 1.99$ (similarly, it painted over with color). Also interesting is the case when $N \rightarrow \infty$.

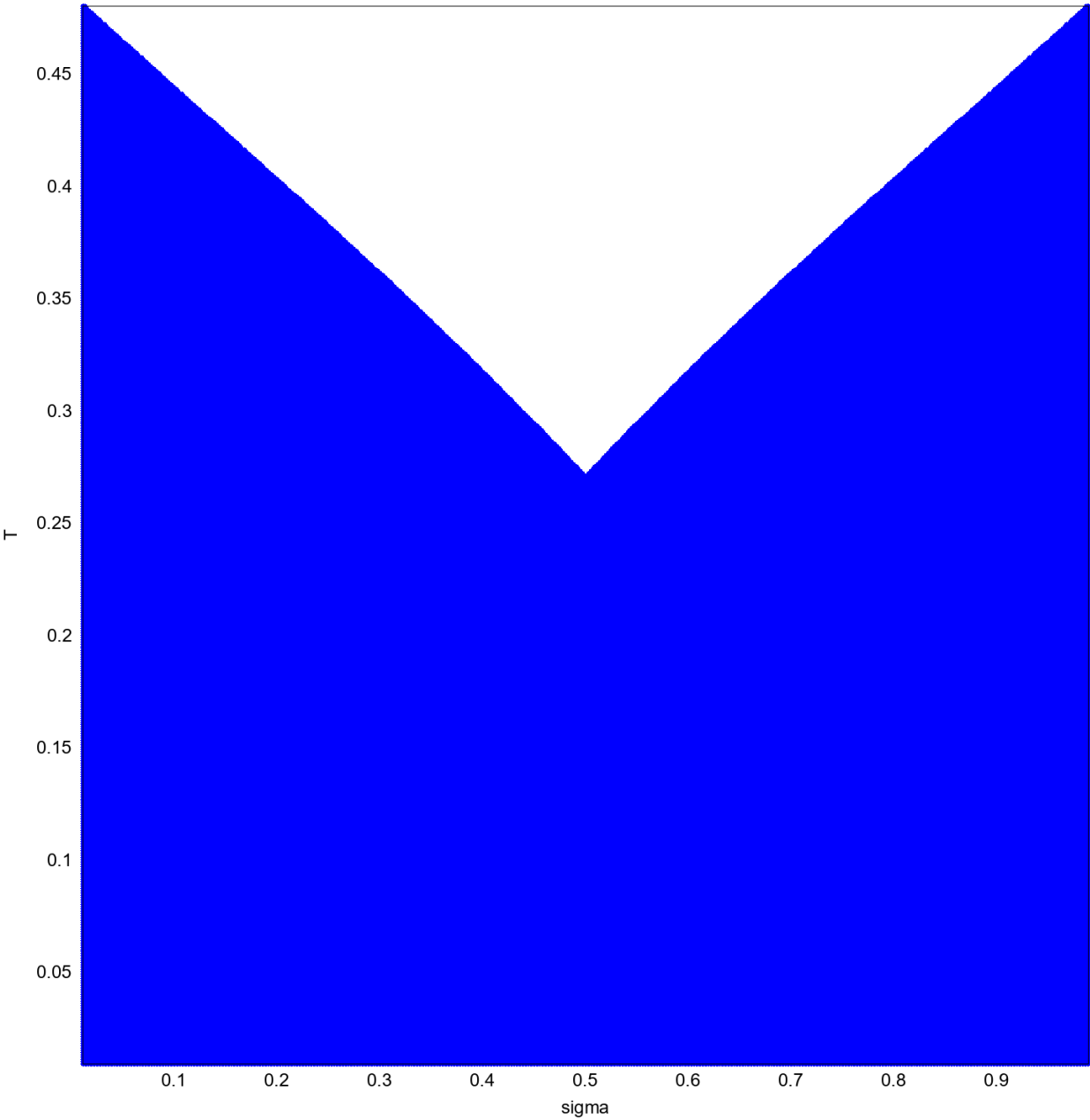


Fig. 1. The region Θ_R for the function $h_1(\sigma, a)$

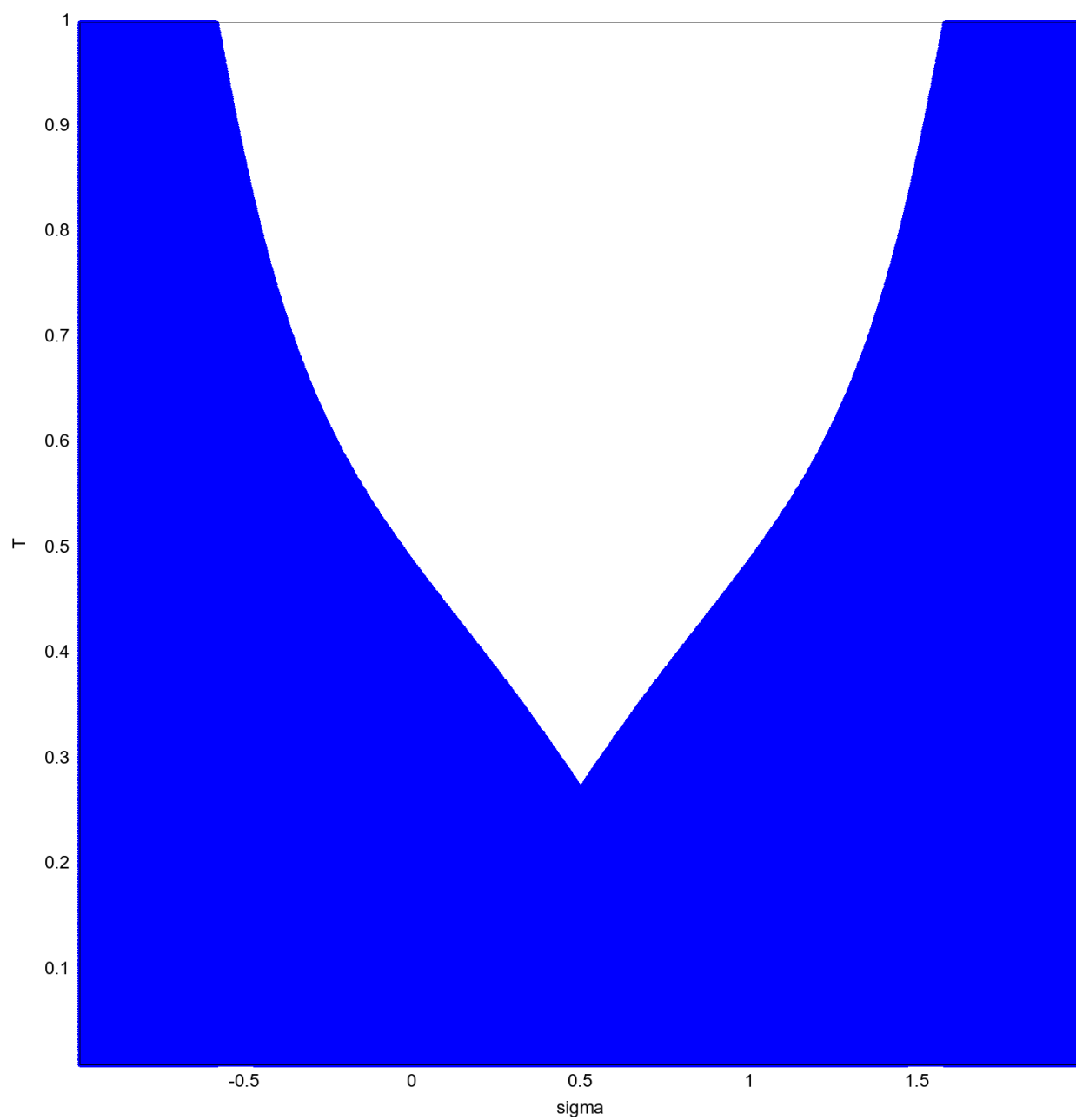


Fig. 2. The region Θ_R for the function $h_2(\sigma, a; 1)$

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