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On a spectral problem for a coupled system of second-order ordinary differential equations with nonlocal weighted integral boundary conditions

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Abstract. In this work, we investigate a spectral problem for a system of second-order ordinary differential equations with a complex spectral parameter subject to weighted integral boundary conditions involving the first derivatives of the unknown functions. For sufficiently large values of the spectral parameter, we derive a priori estimates for solutions of the system and establish the Fredholm solvability of the corresponding operator. The obtained results extend known spectral and solvability properties for ordinary differential operators with nonlocal integral conditions to coupled systems.

Keywords: systems of ordinary differential equations, integral boundary conditions, Fredholm operators, a priori estimates

Mathematics Subject Classification: 34B05, 34B10, 34L10, 47A53.

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О спектральной задаче для связанной системы обыкновенных дифференциальных уравнений второго порядка с нелокальными взвешенными интегральными краевыми условиями

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Аннотация. В данной работе исследуется спектральная задача для системы обыкновенных дифференциальных уравнений второго порядка с комплексным спектральным параметром при взвешенных интегральных краевых условиях, содержащих первые производные неизвестных функций. Для достаточно больших значений спектрального параметра получены априорные оценки решений системы и установлена фредгольмова разрешимость соответствующего оператора. Полученные результаты обобщают известные спектральные свойства и свойства разрешимости обыкновенных дифференциальных операторов с нелокальными интегральными условиями на случай связанных систем.

Ключевые слова: системы обыкновенных дифференциальных уравнений, интегральные краевые условия, фредгольмовы операторы, априорные оценки

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Introduction

Ordinary differential equations with nonlocal and integral boundary conditions have attracted significant attention due to their theoretical importance and applications in physics, hydrodynamics, and diffusion processes; see, for example [1, 2]. One of the fundamental problems in this area is to determine conditions on the coefficients and nonlocal terms under which the spectrum of the corresponding differential operator consists of isolated eigenvalues of finite algebraic multiplicity, usually referred to as a discrete spectrum (see [3]). Early investigations of differential operators with integral conditions and their spectral properties were carried out by Sommerfeld [4] and Picone [5].

In recent decades, second-order differential operators with integral boundary conditions have been studied extensively; see, for instance, [1, 2, 6]. We can also mention the works of S. I. Mitrokhin on the spectral properties of even-order differential operators for systems [7]. In particular, A. L. Skubachevsky [1] obtained a priori estimates for solutions for sufficiently large values of the spectral parameter, established the sectorial structure and discreteness of the spectrum, and proved the Fredholm solvability of the corresponding problems. Furthermore, K. A. Darovskaya and A. L. Skubachevsky [2] considered a second-order ordinary differential equation with integral conditions involving the first derivative of the unknown function. For this problem, the authors derived a priori estimates, described the spectral properties of the associated operator, and established Fredholm solvability. Most of the aforementioned results concern a single differential equation. However, many applied problems naturally lead to systems of differential equations, especially when modeling coupled processes. Although the general strategy of the analysis follows the approach developed for a single equation, the coupling terms in the system through both first-order derivatives and zero-order interactions require additional arguments. Since the integral boundary conditions are independent for each unknown function, the main difficulty arises from the system itself rather than the boundary conditions. Consequently, the derivation of a priori estimates cannot be obtained by a direct componentwise application of the scalar results, and a separate treatment of the system is necessary.

In this paper, we study a system of second-order ordinary differential equations with a spectral parameter given by

$$\begin{aligned} -a_0(t)u''(t) + a_{11}(t)u'(t) + a_{21}v'(t) + a_{31}(t)(u(t) + v(t)) + \lambda^2 u(t) &= f_{01}(t), \\ -a_0(t)v''(t) + a_{12}(t)v'(t) + a_{22}u'(t) + a_{32}(t)(u(t) + v(t)) + \lambda^2 v(t) &= f_{02}(t), \end{aligned} \quad t \in (0, 1), \quad (0.1)$$

subject to the integral conditions

$$\begin{aligned} C_k^1 u &= \int_0^1 \varphi_{k1}(t)u'(t)dt = f_{k1}, \\ C_k^2 v &= \int_0^1 \varphi_{k2}(t)v'(t)dt = f_{k2}, \end{aligned} \quad k = 1, 2. \quad (0.2)$$

Here, a_0, a_{ij} , $i = 1, 2, 3$, $j = 1, 2$, are real-valued functions with $a_0(t) \geq k > 0$, $t \in [0, 1]$, and $a_{ij} \in C[0, 1]$, $f_{0j} \in L^2(0, 1)$, $j = 1, 2$, are complex-valued functions, $f_{k1}, f_{k2} \in \mathbb{C}$, $k = 1, 2$, are constants, $\lambda \in \mathbb{C}$ is a spectral parameter; φ_{11} , φ_{21} and φ_{12} , φ_{22} are linearly independent real-valued functions.

We obtain a priori estimates for solutions in appropriate Sobolev-type norms for sufficiently large values of the spectral parameter, and we establish the Fredholm solvability of the considered system. These results extend the classical theory developed for a single equation to coupled

systems and provide a basis for further investigations of spectral properties and solvability of systems with general integral conditions.

The paper is organized as follows. In Section 1, we introduce some preliminaries and formulate the main result. Section 2 is devoted to the study of auxiliary problems. The proof of the main result is presented in Section 3.

1 Preliminaries and main result

We begin by introducing some sets, spaces, and operators that will be required in the formulation of the main results.

To specify the assumptions imposed on a_0 and on the functions φ_{k1} , φ_{k2} , we introduce some notation.

Let $C^\alpha[a, b]$ denote the Hölder space consisting of all continuous functions $w(t)$, $t \in [a, b]$, for which

$$|w(t_1) - w(t_2)| \leq c |t_1 - t_2|^\alpha, \quad t_1, t_2 \in [a, b],$$

with $c > 0$ and $\alpha \in (0, 1]$ independent of t_1 and t_2 . By $W_\infty^1(a, b)$ we mean the space of absolutely continuous functions $w(t)$, $t \in [a, b]$, whose w' belongs to $L^\infty(a, b)$. For $0 < \beta < \frac{1}{2}$, define

$$\begin{aligned} C_\beta^\alpha[0, 1] &= \{w \in L^2(0, 1) : w \in C^\alpha[0, \beta], w \in C^\alpha[1 - \beta, 1]\}, \\ W_{\infty, \beta}^1(0, 1) &= \{w \in C[0, 1] : w \in W_\infty^1(0, \beta), w \in W_\infty^1(1 - \beta, 1)\}. \end{aligned}$$

Here, it should be noted that the restriction of a function to any subinterval is denoted by the same symbol as the original function. We assume that $a_0 \in W_{\infty, \beta}^1(0, 1)$ and $\varphi_{k1}, \varphi_{k2} \in C_\beta^\alpha[0, 1]$, $\alpha \in (\frac{1}{2}, 1]$. Let $W_2^k(0, 1)$, where $k \in \mathbb{Z}$, $k \geq 0$, denote the Sobolev space of functions in $L^2(0, 1)$ whose generalized derivatives up to order k also belong to $L^2(0, 1)$. In particular, we set $W^2(0, 1) := W_2^2(0, 1)$. The space $W[0, 1] := L^2(0, 1) \times \mathbb{C} \times \mathbb{C}$ is a product space introduced for convenience in the formulation of the problem. In the Sobolev space $W^2(0, 1)$ and in the space $W[0, 1]$, we introduce the following equivalent norms depending on a parameter:

$$\begin{aligned} |||w|||_{W^2(0, 1)} &= \left(\|w\|_{W^2(0, 1)}^2 + |\lambda|^4 \|w\|_{L^2(0, 1)}^2 \right)^{\frac{1}{2}}, \\ |||h_1|||_{W[0, 1]} &= \left(\|h_{01}\|_{L^2(0, 1)}^2 + |\lambda|^3 (|h_{11}|^2 + |h_{21}|^2) \right)^{\frac{1}{2}}, \end{aligned}$$

where $h_1 = (h_{01}, h_{11}, h_{21})$, $|\lambda| \geq 1$. Define $E_1 = W^2(0, 1) \times W^2(0, 1)$, $E_2 = W[0, 1] \times W[0, 1]$, and equip these spaces with the product norms

$$\begin{aligned} |||(u, v)|||_{E_1} &= |||u|||_{W^2(0, 1)} + |||v|||_{W^2(0, 1)}, \\ |||(f_1, f_2)|||_{E_2} &= |||f_1|||_{W[0, 1]} + |||f_2|||_{W[0, 1]}. \end{aligned}$$

On E_1 , consider the bounded linear operator

$$T(\lambda) : E_1 \rightarrow E_2$$

defined by

$$T(\lambda)(u, v) = \left((L_1(u, v) + \lambda^2 u, C_1^1 u, C_2^1 u), (L_2(u, v) + \lambda^2 v, C_1^2 v, C_2^2 v) \right),$$

where

$$\begin{aligned} L_1(u, v) &= -a_0(t)u''(t) + a_{11}(t)u'(t) + a_{21}v'(t) + a_{31}(t)(u(t) + v(t)), \\ L_2(u, v) &= -a_0(t)v''(t) + a_{12}(t)v'(t) + a_{22}u'(t) + a_{32}(t)(u(t) + v(t)). \end{aligned}$$

We next define the unbounded operator

$$\mathcal{L}_C : \mathcal{D}(\mathcal{L}_C) \subset L^2(0, 1) \times L^2(0, 1) \rightarrow L^2(0, 1) \times L^2(0, 1)$$

with

$$\mathcal{D}(\mathcal{L}_C) = \{(u, v) \in E_1 : C_k^1 u = C_k^2 v = 0, k = 1, 2\},$$

by $\mathcal{L}_C(u, v) = (L_1(u, v), L_2(u, v))$.

Further, define the set $\omega_{\varepsilon, q}$ and $\delta_\varphi^1, \delta_\varphi^2$ as follows:

$$\delta_\varphi^1 = \begin{vmatrix} \varphi_{11}(0) & \varphi_{11}(1) \\ \varphi_{21}(0) & \varphi_{21}(1) \end{vmatrix}, \quad \delta_\varphi^2 = \begin{vmatrix} \varphi_{12}(0) & \varphi_{12}(1) \\ \varphi_{22}(0) & \varphi_{22}(1) \end{vmatrix},$$

$$\omega_{\varepsilon, q} = \{\lambda \in \mathbb{C} : |\arg \lambda| \leq \varepsilon \text{ or } |\arg \lambda - \pi| \leq \varepsilon\} \cap \{\lambda \in \mathbb{C} : |\lambda| \geq q\},$$

where $\varepsilon \in (0, \frac{\pi}{2})$ and $q > 1$.

Theorem 1.1. *Let $\delta_\varphi^1 \neq 0$ and $\delta_\varphi^2 \neq 0$. Then, for any $\varepsilon \in (0, \frac{\pi}{2})$, there is $q > 1$ such that for $\lambda \in \omega_{\varepsilon, q}$ every $(u, v) \in E_1$ satisfies the inequality*

$$|||(u, v)|||_{E_1} \leq C |\lambda|^{\frac{1}{2}} |||T(\lambda)(u, v)|||_{E_2}, \quad (1.1)$$

where $C > 0$ does not depend on λ, u and v .

It follows from this theorem that the following result holds.

Corollary 1.1. *Let $\delta_\varphi^1 \neq 0$ and $\delta_\varphi^2 \neq 0$. Then*

(a) *The operators*

$$T(\lambda) : E_1 \rightarrow E_2 \quad \text{and} \quad \mathcal{L}_C : \mathcal{D}(\mathcal{L}_C) \subset L^2(0, 1) \times L^2(0, 1) \rightarrow L^2(0, 1) \times L^2(0, 1)$$

have the Fredholm property, and $\text{ind } T(\lambda) = \text{ind } \mathcal{L}_C = 0$ for all $\lambda \in \mathbb{C}$.

(b) *The spectrum $\sigma(\mathcal{L}_C)$ is discrete.*

(c) *For $\mu \notin \sigma(\mathcal{L}_C)$, the resolvent*

$$R(\mu, \mathcal{L}_C) = (\mu I - \mathcal{L}_C)^{-1} : L^2(0, 1) \times L^2(0, 1) \rightarrow L^2(0, 1) \times L^2(0, 1)$$

is a compact operator.

(d) *For any $0 < \delta < \pi$, all points of the spectrum $\sigma(\mathcal{L}_C)$, except possibly finitely many of them, belong to the angle $|\arg \mu| < \delta$ of the complex plane.*

P r o o f. 1. Consider problem (0.1), (0.2) for $(f_1, f_2) = ((f_{01}, f_{11}, f_{21}), (f_{02}, f_{12}, f_{22})) = (0, 0)$ and $\lambda \in \omega_{\varepsilon, q}$. Then inequality (1.1) yields $(u, v) = (0, 0)$. Hence, the eigenvalues of $\mathcal{L}_C$ do not belong to the set

$$\Omega_{\delta, r} = \{\mu \in \mathbb{C} : |\arg \mu| \geq \delta, |\mu| \geq r\},$$

where $\delta = \pi - 2\varepsilon$ and $r = q^2$.

Clearly, if $\mu = -\lambda^2$ is not an eigenvalue of $\mathcal{L}_C$, the nonhomogeneous problem (0.1), (0.2) admits a unique solution for every $(f_1, f_2) \in E_2$. Combining this observation with inequality (1.1), we conclude that the operator

$$T(\lambda) : E_1 \rightarrow E_2$$

possesses a bounded inverse

$$T^{-1}(\lambda) : E_2 \rightarrow E_1$$

for $\lambda \in \omega_{\varepsilon, q}$. Consequently, the operator

$$\mu I - \mathcal{L}_C : \mathcal{D}(\mathcal{L}_C) \subset L^2(0, 1) \times L^2(0, 1) \rightarrow L^2(0, 1) \times L^2(0, 1)$$

has a bounded inverse

$$(\mu I - \mathcal{L}_C)^{-1} : L^2(0, 1) \times L^2(0, 1) \rightarrow E_1$$

for $\mu \in \Omega_{\delta, r}$, where I is the identity operator in E_1 . Since the embedding $W^2(0, 1) \hookrightarrow L^2(0, 1)$ is compact, it follows from the theorem about a compact resolvent (see [3, Chap. 3, Sec. 6]) that the spectrum $\sigma(\mathcal{L}_C)$ is discrete. Note that

$$\sigma(\mathcal{L}_C) \subset \{\mu \in \mathbb{C} : |\arg \mu| < \delta\} \cup \{\mu \in \mathbb{C} : |\mu| < r\}.$$

2. Let $\lambda_0 \in \omega_{\varepsilon, q}$. For any $\lambda \in \mathbb{C}$, the operator $T(\lambda)$ can be represented as

$$T(\lambda) = [I + (T(\lambda) - T(\lambda_0))T^{-1}(\lambda_0)] T(\lambda_0) \quad (1.2)$$

where I is the identity operator in E_2 . Obviously,

$$(T(\lambda) - T(\lambda_0))(u, v) = ((\lambda^2 - \lambda_0^2)u, 0, 0), ((\lambda^2 - \lambda_0^2)v, 0, 0)).$$

Due to the compact embedding of $W^2(0, 1)$ into $L^2(0, 1)$, the operator

$$(T(\lambda) - T(\lambda_0))T^{-1}(\lambda_0) : E_2 \rightarrow E_2$$

is compact. Therefore, relation (1.2), together with [8, Theorem 12.2], implies that

$$T(\lambda) : E_1 \rightarrow E_2$$

is a Fredholm operator and $\text{ind } T(\lambda) = 0$.

Similarly, one can also verify that the operator

$$\mathcal{L}_C : \mathcal{D}(\mathcal{L}_C) \subset L^2(0, 1) \times L^2(0, 1) \rightarrow L^2(0, 1) \times L^2(0, 1)$$

is of Fredholm type and $\text{ind } \mathcal{L}_C = 0$. □

2 Solvability of the Model Problems

To establish Theorem 1.1, we begin by examining two auxiliary problems.

Problem I

$$L_0 u + \lambda^2 u = -pu''(t) + \lambda^2 u(t) = f_{01}(t), \quad t \in (0, 1), \quad (2.1)$$

$$\varphi_{k1}(0) \int_0^{\frac{1}{2}} u'(t) dt + \varphi_{k1}(1) \int_{\frac{1}{2}}^1 u'(t) dt = f_{k1}, \quad k = 1, 2, \quad (2.2)$$

where $p > 0$ is a constant, $f_{01} \in L^2(0, 1)$ and $f_{k1} \in \mathbb{C}$.

Lemma 2.1. *Let $\delta_\varphi^1 \neq 0$. Then, for every $\varepsilon \in (0, \frac{\pi}{2})$, there exists $q_{01} > 1$ such that for each $\lambda \in \omega_{\varepsilon, q_{01}}$, every solution of problem (2.1), (2.2) $u \in W^2(0, 1)$ satisfies*

$$|||u|||_{W^2(0,1)} \leq K_1 |||f_1|||_{W[0,1]}, \quad (2.3)$$

where $f_1 = (f_{01}, f_{11}, f_{21})$ and $K_1 > 0$ is independent of λ and u .

P r o o f. 1. The problem (2.1), (2.2) may be rewritten equivalently as

$$L_0 u + \lambda^2 u = -pu''(t) + \lambda^2 u(t) = f_{01}(t), \quad t \in (0, 1), \quad (2.4)$$

$$\varphi_{k1}(0) \left[u\left(\frac{1}{2}\right) - u(0) \right] + \varphi_{k1}(1) \left[u(1) - u\left(\frac{1}{2}\right) \right] = f_{k1}, \quad k = 1, 2. \quad (2.5)$$

From conditions (2.5), we obtain the following system of linear algebraic equations for $u(0)$ and $u(1)$:

$$\begin{cases} -\varphi_{11}(0)u(0) + \varphi_{11}(1)u(1) = f_{11} - (\varphi_{11}(0) - \varphi_{11}(1))u\left(\frac{1}{2}\right), \\ -\varphi_{21}(0)u(0) + \varphi_{21}(1)u(1) = f_{21} - (\varphi_{21}(0) - \varphi_{21}(1))u\left(\frac{1}{2}\right), \end{cases} \quad (2.6)$$

whose determinant is

$$\overline{\delta}_1 = \begin{vmatrix} -\varphi_{11}(0) & \varphi_{11}(1) \\ -\varphi_{21}(0) & \varphi_{21}(1) \end{vmatrix} = -\delta_\varphi^1 \neq 0.$$

Therefore, system (2.6) has a unique solution. Using Cramer's rule, one can express $u(0)$ and $u(1)$ in terms of $\varphi_{k1}(0)$, $\varphi_{k1}(1)$, f_{k1} , $k = 1, 2$, and $u(\frac{1}{2})$, and obtains

$$u(0) = \frac{f_{21}\varphi_{11}(1) - f_{11}\varphi_{21}(1)}{\delta_\varphi^1} + u\left(\frac{1}{2}\right), \quad u(1) = \frac{f_{21}\varphi_{11}(0) - f_{11}\varphi_{21}(0)}{\delta_\varphi^1} + u\left(\frac{1}{2}\right).$$

Thus, the nonlocal boundary conditions (2.5) may be rewritten as

$$\begin{cases} u(0) - u\left(\frac{1}{2}\right) = \widetilde{f}_{11}, \\ u(1) - u\left(\frac{1}{2}\right) = \widetilde{f}_{21}, \end{cases} \quad (2.7)$$

where

$$\widetilde{f}_{k1} = \frac{f_{21}\varphi_{11}(\tau_k) - f_{11}\varphi_{21}(\tau_k)}{\delta_\varphi^1}, \quad k = 1, 2, \quad \tau_1 = 1, \quad \tau_2 = 0.$$

2. Define the operator $T_{01}(\lambda) : W^2(0, 1) \rightarrow W[0, 1]$ by

$$T_{01}(\lambda)u = ((L_0 + \lambda^2 I)u, C_1^{01}u, C_2^{01}u),$$

where

$$C_1^{01}u = u(0) - u\left(\frac{1}{2}\right), \quad C_2^{01}u = u(1) - u\left(\frac{1}{2}\right).$$

Thus, problem (2.4), (2.7) becomes a multi-point boundary value problem. Using Lemma 1.1.1 and Theorem 1.1.1 of [1], we conclude that if $\delta_\varphi^1 \neq 0$, then for every $\varepsilon \in (0, \frac{\pi}{2})$ there exists $q_{01} = q_{01}(\varepsilon) > 1$ such that for $\lambda \in \omega_{\varepsilon, q_{01}}$ the operator $T_{01}(\lambda)$ possesses a bounded inverse

$$T_{01}^{-1}(\lambda) : W[0, 1] \rightarrow W^2(0, 1),$$

and every $u \in W^2(0, 1)$ satisfies

$$|||u|||_{W^2(0,1)} \leq K_1 |||T_{01}(\lambda)u|||_{W[0,1]}, \quad (2.8)$$

where $K_1 > 0$ does not depend on λ and u .

Since problem (2.4), (2.7) is equivalent to problem (2.1), (2.2), the estimate (2.8) implies (2.3). This completes the proof. $\square$

Problem II

$$L_0 v + \lambda^2 v = -p v''(t) + \lambda^2 v(t) = f_{02}(t), \quad t \in (0, 1), \quad (2.9)$$

$$\varphi_{k2}(0) \int_0^{\frac{1}{2}} v'(t) dt + \varphi_{k2}(1) \int_{\frac{1}{2}}^1 v'(t) dt = f_{k2}, \quad k = 1, 2, \quad (2.10)$$

where $p > 0$ is a constant, $f_{02} \in L^2(0, 1)$ and $f_{k2} \in \mathbb{C}$.

Similar to the proof of Lemma 2.1, the following lemma holds.

Lemma 2.2. *Let $\delta_\varphi^2 \neq 0$. Then, for every $\varepsilon \in (0, \frac{\pi}{2})$, there exists $q_{02} > 1$ such that for each $\lambda \in \omega_{\varepsilon, q_{02}}$ every solution of problem (2.9), (2.10) $v \in W^2(0, 1)$ satisfies*

$$|||v|||_{W^2(0,1)} \leq K_2 |||f_2|||_{W[0,1]},$$

where $f_2 = (f_{02}, f_{12}, f_{22})$ and $K_2 > 0$ is independent of λ and v .

3 Proof of Theorem 1.1

1. To begin with, we apply a standard straightening procedure to equations (0.1) in order to ensure that, in sufficiently small neighborhoods of the endpoints 0 and 1, the coefficient in front of the second derivative becomes constant. For this purpose, we introduce a new variable τ by

$$\tau = \frac{1}{d} \int_0^t \frac{dt'}{b(t')},$$

where

$$d = \int_0^1 \frac{dt'}{b(t')}, \quad b(t') = \begin{cases} \sqrt{a_0(t')} & \text{for } t' \in [0, \beta] \cup [1 - \beta, 1], \\ \frac{(1 - t' - \beta)\sqrt{a_0(\beta)} + (t' - \beta)\sqrt{a_0(1 - \beta)}}{1 - 2\beta} & \text{for } t' \in (\beta, 1 - \beta). \end{cases}$$

In view of the assumptions imposed on a_0 , namely $a_0 \in W_{\infty, \beta}^1(0, 1)$ and $a_0(t) \geq k > 0$, we have

$$0 < k_1 \leq \frac{1}{b(t)} \leq k^{-\frac{1}{2}}, \quad \frac{1}{b} \in W_{\infty, \beta}^1(0, 1).$$

Therefore, the function $\tau = \tau(t)$ maps the interval $[0, 1]$ onto itself, and

$$\frac{d\tau}{dt} = \frac{1}{db(t)} \geq k_1 k^{\frac{1}{2}} > 0, \quad \frac{d\tau}{dt} \in W_{\infty, \beta}^1(0, 1).$$

With this change of variables, problem (0.1), (0.2) takes the form

$$\begin{aligned} -\bar{a}_0(\tau) \bar{u}''(\tau) + \bar{a}_{11}(\tau) \bar{u}'(\tau) + \bar{a}_{21}(\tau) \bar{v}'(\tau) + \bar{a}_{31}(\tau) (\bar{u}(\tau) + \bar{v}(\tau)) + \lambda^2 \bar{u}(\tau) &= \bar{f}_{01}(\tau), \\ -\bar{a}_0(\tau) \bar{v}''(\tau) + \bar{a}_{12}(\tau) \bar{v}'(\tau) + \bar{a}_{22}(\tau) \bar{u}'(\tau) + \bar{a}_{32}(\tau) (\bar{u}(\tau) + \bar{v}(\tau)) + \lambda^2 \bar{v}(\tau) &= \bar{f}_{02}(\tau), \end{aligned} \quad \tau \in (0, 1),$$

$$\begin{aligned} \int_0^1 \bar{\varphi}_{k1}(\tau) \bar{u}'(\tau) d\tau &= f_{k1}, \\ \int_0^1 \bar{\varphi}_{k2}(\tau) \bar{v}'(\tau) d\tau &= f_{k2}, \end{aligned} \quad k = 1, 2,$$

where

$$\bar{u}(\tau) = u(t(\tau)), \quad \bar{v}(\tau) = v(t(\tau)), \quad \bar{f}_{0j}(\tau) = f_{0j}(t(\tau)), \quad j = 1, 2,$$

$$\begin{aligned} \bar{a}_0(\tau) &= \frac{a_0(t(\tau))}{d^2 b^2(t(\tau))} \in W_{\infty, \beta_0}^1(0, 1), \\ \bar{a}_{1j}(\tau) &= \frac{a_{1j}(t(\tau))}{db(t(\tau))} + \frac{a_0(t(\tau))b'(t(\tau))}{d^2 b^3(t(\tau))} \in L^\infty(0, 1), \\ \bar{a}_{2j}(\tau) &= \frac{a_{2j}(t(\tau))}{db(t(\tau))} \in L^\infty(0, 1), \\ \bar{a}_{3j}(\tau) &= a_{3j}(t(\tau)) \in C[0, 1], \\ \bar{\varphi}_{k1}(\tau) &= \varphi_{k1}(t(\tau)) \in C_{\beta_0}^\alpha[0, 1], \\ \bar{\varphi}_{k2}(\tau) &= \varphi_{k2}(t(\tau)) \in C_{\beta_0}^\alpha[0, 1], \end{aligned}$$

and

$$\beta_0 = \min \left\{ d^{-1} \int_0^\beta b^{-1}(t') dt', 1 - d^{-1} \int_0^{1-\beta} b^{-1}(t') dt' \right\}.$$

Moreover,

$$\bar{a}_0(\tau) = \frac{1}{d^2}, \quad \tau \in [0, \beta_0] \cup [1 - \beta_0, 1].$$

It is important to note that inequality (1.1) is invariant under the substitution $\tau = \tau(t)$. Furthermore, if $\delta_\varphi^1 \neq 0$ and $\delta_\varphi^2 \neq 0$, then

$$\begin{aligned} \delta_\varphi^1 &= \begin{vmatrix} \bar{\varphi}_{11}(0) & \bar{\varphi}_{11}(1) \\ \bar{\varphi}_{21}(0) & \bar{\varphi}_{21}(1) \end{vmatrix} = \begin{vmatrix} \varphi_{11}(0) & \varphi_{11}(1) \\ \varphi_{21}(0) & \varphi_{21}(1) \end{vmatrix} = \delta_\varphi^1 \neq 0, \\ \delta_\varphi^2 &= \begin{vmatrix} \bar{\varphi}_{12}(0) & \bar{\varphi}_{12}(1) \\ \bar{\varphi}_{22}(0) & \bar{\varphi}_{22}(1) \end{vmatrix} = \begin{vmatrix} \varphi_{12}(0) & \varphi_{12}(1) \\ \varphi_{22}(0) & \varphi_{22}(1) \end{vmatrix} = \delta_\varphi^2 \neq 0. \end{aligned}$$

Therefore, without loss of generality, we may assume that $a_0 \in W_{\infty, \beta}^1(0, 1)$, $a_{ij} \in L^\infty(0, 1)$, $i = 1, 2, 3$, $j = 1, 2$, $\varphi_{k1}, \varphi_{k2} \in C_{\beta}^\alpha[0, 1]$, $k = 1, 2$, $a_0(t) \geq k > 0$, $a_0(t) = a_0(0)$ for $t \in [0, \beta] \cup [1 - \beta, 1]$, $\delta_\varphi^1 \neq 0$ and $\delta_\varphi^2 \neq 0$.

2. As the initial step in proving inequality (1.1), we establish an a priori estimate for solutions of problem (0.1), (0.2) on the closed interval $[\frac{\delta}{3}, 1 - \frac{\delta}{3}]$, where $\delta = \delta(\lambda) > 0$. Let $\xi \in C^\infty(\mathbb{R})$ be a truncating function such that

$$0 \leq \xi(t) \leq 1, \quad \xi(t) = 1 \quad \text{for } |t| < \frac{1}{4}, \quad \xi(t) = 0 \quad \text{for } |t| > \frac{1}{3}.$$

For each $\lambda \in \omega_{\varepsilon, q_1}$, define the function

$$\phi(t) = \xi\left(\frac{t}{\delta}\right) + \xi\left(\frac{t-1}{\delta}\right),$$

where $q_1 = \max\{q_{01}, q_{02}, \beta^{-3}\}$, $\delta = \frac{1}{|\lambda|^{\frac{1}{3}+2\gamma}}$ and $\frac{1-\alpha}{6\alpha+3} < \gamma < \frac{1}{12}$. Observe that the condition $\frac{1}{2} < \alpha \leq 1$ implies the inequality $\frac{1-\alpha}{6\alpha+3} < \frac{1}{12}$.

It is straightforward to verify that

$$|\phi^{(j)}(t)| \leq k_1 \delta^{-j}, \quad t \in \mathbb{R}, \quad j = 1, 2, \quad (3.1)$$

with a constant $k_1 > 0$ independent of t and δ . Since $|\lambda| \geq q_1$, we have $\delta < q_1^{-\frac{1}{3}} \leq \beta$.

Taking into account that the functions a_0 and a_{ij} , $i = 1, 2, 3$, $j = 1, 2$, are bounded, and using inequality (3.1) together with the Leibnitz formula, as well as the a priori estimates for solutions of “local” boundary value problems with a parameter [9], we conclude that for any $\varepsilon > 0$ there exists $q_2 \geq q_1$ such that for all $\lambda \in \omega_{\varepsilon, q_2}$ every solution $(u, v) \in E_1$ of problem (0.1), (0.2) satisfies

$$\begin{aligned}
& |||(1 - \phi)u|||_{W^2(0,1)} + |||(1 - \phi)v|||_{W^2(0,1)} \\
& \leq k_2 \left(\|L_1((1 - \phi)u, v) - (a_{21}v' + a_{31}v) + \lambda^2(1 - \phi)u\|_{L^2(0,1)} \right. \\
& \quad \left. + \|L_2(u, (1 - \phi)v) - (a_{22}u' + a_{32}u) + \lambda^2(1 - \phi)v\|_{L^2(0,1)} \right) \\
& \leq k_3 \left(\|L_1(u, v) - (a_{21}v' + a_{31}v) + \lambda^2u\|_{L^2(0,1)} + |\lambda|^{\frac{2}{3}+4\gamma} \|u\|_{L^2(0,1)} \right. \\
& \quad \left. + |\lambda|^{\frac{1}{3}+2\gamma} (\|u\|_{L^2(0,1)} + \|u'\|_{L^2(0,1)}) + \|L_2(u, v) - (a_{22}u' + a_{32}u) + \lambda^2v\|_{L^2(0,1)} \right. \\
& \quad \left. + |\lambda|^{\frac{2}{3}+4\gamma} \|v\|_{L^2(0,1)} + |\lambda|^{\frac{1}{3}+2\gamma} (\|v\|_{L^2(0,1)} + \|v'\|_{L^2(0,1)}) \right) \\
& \leq k_4 \left(\|L_1(u, v) + \lambda^2u\|_{L^2(0,1)} + |\lambda|^{\frac{2}{3}+4\gamma} \|u\|_{L^2(0,1)} + |\lambda|^{\frac{1}{3}+2\gamma} \|u\|_{W^1(0,1)} + \|v\|_{W^1(0,1)} \right. \\
& \quad \left. + \|L_2(u, v) + \lambda^2v\|_{L^2(0,1)} + |\lambda|^{\frac{2}{3}+4\gamma} \|v\|_{L^2(0,1)} + |\lambda|^{\frac{1}{3}+2\gamma} \|v\|_{W^1(0,1)} + \|u\|_{W^1(0,1)} \right) \\
& \leq k_4 \left(\|L_1(u, v) + \lambda^2u\|_{L^2(0,1)} + |\lambda|^{\frac{2}{3}+4\gamma} \|u\|_{L^2(0,1)} + |\lambda|^{\frac{1}{3}+2\gamma} \|u\|_{W^1(0,1)} \right. \\
& \quad \left. + \|L_2(u, v) + \lambda^2v\|_{L^2(0,1)} + |\lambda|^{\frac{2}{3}+4\gamma} \|v\|_{L^2(0,1)} + |\lambda|^{\frac{1}{3}+2\gamma} \|v\|_{W^1(0,1)} \right).
\end{aligned}$$

Here and further $k_2, \dots, k_7 > 0$ do not depend on λ and u .

Using interpolation inequality (see [9]), which states that for any $w \in W_2^k(a, b)$ and $\lambda \in \mathbb{C}$

$$|\lambda|^{k-s} \|w\|_{W_2^s(a,b)} \leq c \left(\|w\|_{W_2^k(a,b)} + |\lambda|^k \|w\|_{L^2(a,b)} \right), \quad (3.2)$$

where $k, s \in \mathbb{N}$, $0 < s < k$, and $c > 0$ is independent of w and λ , we get

$$\begin{aligned}
& |||(1 - \phi)u|||_{W^2(0,1)} + |||(1 - \phi)v|||_{W^2(0,1)} \\
& \leq k_5 \left(\|L_1(u, v) + \lambda^2u\|_{L^2(0,1)} + \|L_2(u, v) + \lambda^2v\|_{L^2(0,1)} \right. \\
& \quad \left. + |\lambda|^{-\frac{2}{3}+2\gamma} (\|u\|_{W^2(0,1)} + \|v\|_{W^2(0,1)}) + \left(|\lambda|^{\frac{2}{3}+4\gamma} + |\lambda|^{\frac{4}{3}+2\gamma} \right) (\|u\|_{L^2(0,1)} + \|v\|_{L^2(0,1)}) \right) \\
& \leq k_6 \left(\|L_1(u, v) + \lambda^2u\|_{L^2(0,1)} + |\lambda|^{-\frac{2}{3}+2\gamma} (\|u\|_{W^2(0,1)} + |\lambda|^2 \|u\|_{L^2(0,1)}) \right. \\
& \quad \left. + \|L_2(u, v) + \lambda^2v\|_{L^2(0,1)} + |\lambda|^{-\frac{2}{3}+2\gamma} (\|v\|_{W^2(0,1)} + |\lambda|^2 \|v\|_{L^2(0,1)}) \right).
\end{aligned}$$

Consequently, one deduces

$$|||(1 - \phi)(u, v)|||_{E_1} \leq k_7 (|||(f_1, f_2)|||_{E_2} + |\lambda|^{-\sigma} |||(u, v)|||_{E_1}), \quad (3.3)$$

where $\sigma = \frac{2}{3} - 2\gamma > 0$.

3. We now turn to deriving an a priori estimate for solutions of problem (0.1), (0.2) in neighborhoods of the endpoints of the interval $(0, 1)$. To this end, we introduce the operators L_0 and L_{11} via the formulas $L_0u = -a_0(0)u''(t)$ and $L_{11}u = -a_0(t)u''(t)$, $t \in (0, 1)$. Setting

$p = a_0(0)$ in (2.1), Lemma 2.1 yields

$$\begin{aligned} & |||\phi u|||_{W^2(0,1)} \\ & \leq K_1 \left(\|(L_0 + \lambda^2 I)(\phi u)\|_{L^2(0,1)} + |\lambda|^{\frac{3}{2}} \sum_{k=1}^2 \left| \varphi_{k1}(0) \int_0^{\frac{1}{2}} (\phi(t)u(t))' dt + \varphi_{k1}(1) \int_{\frac{1}{2}}^1 (\phi(t)u(t))' dt \right| \right). \end{aligned} \quad (3.4)$$

Proceeding in the same manner as in Part 2 of the proof, we arrive at

$$\begin{aligned} & \|(L_0 + \lambda^2 I)(\phi u)\|_{L^2(0,1)} \\ & \leq k_8 \left(\|L_1(u, v) + \lambda^2 u\|_{L^2(0,1)} + |\lambda|^{-\sigma} |||(u, v)|||_{E_1} + \|(L_{11} - L_0)(\phi u)\|_{L^2(0,1)} \right). \end{aligned} \quad (3.5)$$

It follows from the definition of the function $\phi(t)$ and the inequality $\delta < \beta$ that $\text{supp } \phi \subset [0, \beta] \cup [1 - \beta, 1]$. Then, since $a_0(t) = a_0(0)$ for $t \in [0, \beta] \cup [1 - \beta, 1]$, we have

$$\|(L_{11} - L_0)(\phi u)\|_{L^2(0,1)} = 0.$$

Further, we obtain

$$\begin{aligned} & |\lambda|^{\frac{3}{2}} \left| \varphi_{k1}(0) \int_0^{\frac{1}{2}} (\phi(t)u(t))' dt + \varphi_{k1}(1) \int_{\frac{1}{2}}^1 (\phi(t)u(t))' dt \right| \\ & \leq |\lambda|^{\frac{3}{2}} \left| \varphi_{k1}(0) \int_0^{\frac{1}{2}} \phi(t)u'(t) dt + \varphi_{k1}(1) \int_{\frac{1}{2}}^1 \phi(t)u'(t) dt \right| + k_9 |\lambda|^{\frac{11}{6}+2\gamma} \|u\|_{L^2(0,1)} \\ & \leq |\lambda|^{\frac{3}{2}} \left| \int_0^1 \varphi_{k1}(t)u'(t) dt \right| + k_9 |\lambda|^{-\frac{1}{6}+2\gamma} |\lambda|^2 \|u\|_{L^2(0,1)} + |\lambda|^{\frac{3}{2}} (|I_{k1}| + |I_{k2}| + |I_{k3}|). \end{aligned} \quad (3.6)$$

Where

$$\begin{aligned} I_{k1} &= \int_0^1 \varphi_{k1}(t) (\phi(t) - 1) u'(t) dt, \\ I_{k2} &= \int_0^{\frac{1}{2}} (\varphi_{k1}(0) - \varphi_{k1}(t)) \phi(t) u'(t) dt, \quad I_{k3} = \int_{\frac{1}{2}}^1 (\varphi_{k1}(1) - \varphi_{k1}(t)) \phi(t) u'(t) dt. \end{aligned}$$

Combining (3.4)–(3.6), we obtain

$$\begin{aligned} & |||\phi u|||_{W^2(0,1)} \\ & \leq k_{10} \left(|||f_1|||_{W[0,1]} + |\lambda|^{-\sigma} |||(u, v)|||_{E_1} + |\lambda|^{-\frac{1}{6}+2\gamma} |||u|||_{W^2(0,1)} + |\lambda|^{\frac{3}{2}} \sum_{k=1}^2 (|I_{k1}| + |I_{k2}| + |I_{k3}|) \right). \end{aligned}$$

Let us estimate the integrals I_{kj} , $j = 1, 2, 3$.

To estimate I_{k1} we introduce a new truncating function $\xi_1 \in C^\infty(\mathbb{R})$ such that

$$0 \leq \xi_1(t) \leq 1, \quad \xi_1(t) = 1 \text{ for } |t| < \frac{1}{8}, \quad \xi_1(t) = 0 \text{ for } |t| > \frac{1}{4}.$$

For each $\lambda \in \omega_{\varepsilon, q_2}$, consider the function

$$\phi_1(t) = \xi_1\left(\frac{t}{\delta}\right) + \xi_1\left(\frac{t-1}{\delta}\right).$$

Observe analogously to (3.1) that

$$\left| \phi_1^{(j)}(t) \right| \leq k_{11} \delta^{-j}, \quad t \in \mathbb{R}, \quad j = 1, 2. \quad (3.7)$$

Taking into account the inequalities (3.2) and (3.7), the Leibnitz formula, and the a priori estimates for solutions of “local” boundary value problems with a parameter, and proceeding as in Part 2 of the proof, we find that

$$\begin{aligned} |\lambda|^{\frac{3}{2}} |I_{k1}| &\leq k_{12} |\lambda|^{\frac{3}{2}} \|u'\|_{L^2(\frac{\delta}{4}, 1-\frac{\delta}{4})} \leq k_{12} |\lambda|^{\frac{3}{2}} \|((1-\phi_1)u)'\|_{L^2(0,1)} \\ &\leq k_{13} |\lambda|^{\frac{1}{2}} |\lambda| \|(1-\phi_1)u\|_{W_2^1(0,1)} \leq k_{13} |\lambda|^{\frac{1}{2}} \left(\|(1-\phi_1)u\|_{W^2(0,1)} + |\lambda|^2 \|(1-\phi_1)u\|_{L^2(0,1)} \right) \\ &\leq k_{14} |\lambda|^{\frac{1}{2}} \|L_1((1-\phi_1)u, v) - (a_{21}v' + a_{31}v) + \lambda^2(1-\phi_1)u\|_{L^2(0,1)} \\ &\leq k_{15} |\lambda|^{\frac{1}{2}} \left(\|f_1\|_{W[0,1]} + |\lambda|^{-\sigma} \|(u, v)\|_{E_1} \right). \end{aligned}$$

Since $(\text{supp } \phi) \cap [0, \frac{1}{2}] \subset (0, \frac{\delta}{3})$, $\varphi_{k1} \in C^\alpha[0, \delta]$, and using (3.2), we get

$$\begin{aligned} |\lambda|^{\frac{3}{2}} |I_{k2}| &\leq |\lambda|^{\frac{3}{2}} \left(\int_0^{\frac{\delta}{3}} |\varphi_{k1}(0) - \varphi_{k1}(t)|^2 dt \right)^{\frac{1}{2}} \|u'\|_{L^2(0,1)} \leq k_{16} |\lambda|^{\frac{3}{2}} \left(\int_0^{\frac{\delta}{3}} t^{2\alpha} dt \right)^{\frac{1}{2}} \|u'\|_{L^2(0,1)} \\ &\leq k_{17} |\lambda|^{\frac{3}{2}} (\delta^{2\alpha+1})^{\frac{1}{2}} \|u'\|_{L^2(0,1)} \leq k_{17} |\lambda|^{\frac{3}{2} - (\frac{1}{3}+2\gamma)(\frac{1}{2}+\alpha)} \|u\|_{W_2^1(0,1)} \\ &\leq k_{18} |\lambda|^{\frac{1}{2} - (\frac{1}{3}+2\gamma)(\frac{1}{2}+\alpha)} \left(\|u\|_{W^2(0,1)} + |\lambda|^2 \|u\|_{L^2(0,1)} \right) \\ &\leq k_{18} |\lambda|^{\frac{1}{2} - (\frac{1}{3}+2\gamma)(\frac{1}{2}+\alpha)} \|u\|_{W^2(0,1)}. \end{aligned}$$

Similarly,

$$|\lambda|^{\frac{3}{2}} |I_{k3}| \leq k_{19} |\lambda|^{\frac{1}{2} - (\frac{1}{3}+2\gamma)(\frac{1}{2}+\alpha)} \|u\|_{W^2(0,1)}.$$

Combining the preceding estimates, we conclude that

$$\begin{aligned} |\lambda|^{\frac{1}{2}} (|I_{k1}| + |I_{k2}| + |I_{k3}|) &\leq k_{20} \left(|\lambda|^{\frac{1}{2}} \|f_1\|_{W[0,1]} + |\lambda|^{\frac{1}{2}-\sigma} \|(u, v)\|_{E_1} + |\lambda|^{\frac{1}{2} - (\frac{1}{3}+2\gamma)(\frac{1}{2}+\alpha)} \|u\|_{W^2(0,1)} \right). \end{aligned}$$

Consequently,

$$\begin{aligned} \| \phi u \|_{W^2(0,1)} &\leq k_{21} \left(\|f_1\|_{W[0,1]} + |\lambda|^{-\frac{1}{6}+2\gamma} \|u\|_{W^2(0,1)} + |\lambda|^{\frac{1}{2}} \|f_1\|_{W[0,1]} \right. \\ &\quad \left. + |\lambda|^{\frac{1}{2}-\sigma} \|(u, v)\|_{E_1} + |\lambda|^{\frac{1}{2} - (\frac{1}{3}+2\gamma)(\frac{1}{2}+\alpha)} \|u\|_{W^2(0,1)} \right) \\ &\leq k_{22} \left(|\lambda|^{\frac{1}{2}} \|f_1\|_{W[0,1]} + |\lambda|^{-\frac{1}{6}+2\gamma} \|u\|_{W^2(0,1)} \right. \\ &\quad \left. + |\lambda|^{\frac{1}{2}-\sigma} \|(u, v)\|_{E_1} + |\lambda|^{\frac{1}{2} - (\frac{1}{3}+2\gamma)(\frac{1}{2}+\alpha)} \|u\|_{W^2(0,1)} \right). \quad (3.8) \end{aligned}$$

Now, setting $p = a_0(0)$ in (2.9). Then, from Lemma 2.2 and similarly to the proof of (3.8), one has

$$\begin{aligned} |||\phi v|||_{W^2(0,1)} &\leq k_{23} \left(|\lambda|^{\frac{1}{2}} |||f_2|||_{W[0,1]} + |\lambda|^{-\frac{1}{6}+2\gamma} |||v|||_{W^2(0,1)} \right. \\ &\quad \left. + |\lambda|^{\frac{1}{2}-\sigma} |||(u, v)|||_{E_1} + |\lambda|^{\frac{1}{2}-(\frac{1}{3}+2\gamma)(\frac{1}{2}+\alpha)} |||v|||_{W^2(0,1)} \right). \end{aligned} \quad (3.9)$$

It follows from (3.8) and (3.9) that

$$\begin{aligned} |||\phi(u, v)|||_{E_1} &\leq k_{24} \left(|\lambda|^{\frac{1}{2}} |||(f_1, f_2)|||_{E_2} + |\lambda|^{-\frac{1}{6}+2\gamma} |||(u, v)|||_{E_1} \right. \\ &\quad \left. + |\lambda|^{\frac{1}{2}-\sigma} |||(u, v)|||_{E_1} + |\lambda|^{\frac{1}{2}-(\frac{1}{3}+2\gamma)(\frac{1}{2}+\alpha)} |||(u, v)|||_{E_1} \right). \end{aligned}$$

Note that $\frac{1}{2} - \sigma = -\frac{1}{6} + 2\gamma$. Then, for $|||\phi(u, v)|||_{E_1}$ we have the following estimate

$$|||\phi(u, v)|||_{E_1} \leq k_{25} \left(|\lambda|^{\frac{1}{2}} |||(f_1, f_2)|||_{E_2} + |\lambda|^{-\chi} |||(u, v)|||_{E_1} \right), \quad (3.10)$$

where

$$\chi = \min \left\{ \frac{1}{6} - 2\gamma, -\frac{1}{2} + \left(\frac{1}{3} + 2\gamma \right) \left(\frac{1}{2} + \alpha \right) \right\}.$$

The condition

$$\frac{1 - \alpha}{6\alpha + 3} < \gamma < \frac{1}{12}$$

implies that $\chi > 0$.

From (3.3) and (3.10), we get

$$|||(u, v)|||_{E_1} \leq k_{26} \left(|\lambda|^{\frac{1}{2}} |||(f_1, f_2)|||_{E_2} + |\lambda|^{-\chi} |||(u, v)|||_{E_1} \right). \quad (3.11)$$

Therefore, we derive inequality (1.1) from (3.11) by choosing $q > q_2$ so that $k_{26}q^{-\chi} < 1$. The proof is complete.

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